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ON QUADRATICALLY HYPONORMAL WEIGHTED SHIFTS

Mi Ryeong Lee*†

Department of Mathematics, Kyungpook National University,
Daegu 702-701, Korea
E-mail: leemr@knu.ac.kr

Abstract

The positive quadratic hyponormality are characterized and quadratic hyponormality of certain such backstep extensions of arbitrary length are generalized from some earlier results. These results can be applied to the positive quadratic hyponormality of the weighted shift W_α associated to the weight sequence $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. This yields the set of positive real numbers x such that W_α is quadratically hyponormal for some u, v and w, an open problem in [9], and one produces an interval in x with nonempty interior in the positive real line for quadratic hyponormality but not positive quadratic hyponormality for such a weighted shift W_α.

1. Introduction. This is based on the joint work with George R. Exner, Il Bong Jung, and Sun Hyun Park and was talked at the 2011 RIMS symposium: Structural study of operators via spectra or numerical ranges, which was held at Kyoto University on November 14-16 in 2011. And also this will be appeared in some other journal as a version with detail proofs and additional results (cf. [20]).

Let $\mathcal{H}$ be a separable complex Hilbert space and $L(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. An operator T in $L(\mathcal{H})$ is *normal* if it commutes with its adjoint, *subnormal* if it is the restriction of a normal operator to an invariant subspace, and *hyponormal* if $T^*T \geq TT^*$. For A, $B \in L(\mathcal{H})$, we set $[A, B] := AB - BA$. A k-tuple $T = (T_1, \cdots, T_k)$ of operators in $L(\mathcal{H})$ is called *hyponormal* if the operator matrix $([T_j^*, T_i])_{i,j=1}^k$ is positive on the direct sum of k copies of $\mathcal{H}$. For $k \in \mathbb{N}$ and $T \in L(\mathcal{H})$, T is said to be *k-hyponormal* if $(I, T, \cdots, T^k)$ is hyponormal. It is well-known that $T \in L(\mathcal{H})$ is subnormal if and only if T is k-hyponormal for all $k \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers ([14], [2]). A k-tuple $T = (T_1, \cdots, T_k)$ is *weakly hyponormal* if $\lambda_1 T_1 + \cdots + \lambda_k T_k$ is hyponormal for every $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, k$, where $\mathbb{C}$ is the set of complex numbers. An operator T is *weakly k-hyponormal* if $(T, T^2, \cdots, T^k)$ is weakly hyponormal; equivalently, for every complex polynomial p of degree k or less, $p(T)$ is hyponormal ([4]). An operator T is *polynomially hyponormal* if, for every polynomial p with complex coefficients, $p(T)$ is hyponormal. In [4] Curto initiated the study of classes (actually or potentially) between hyponormal and subnormal with the implications: "subnormal $\Rightarrow \cdots \Rightarrow$ 2-hyponormal $\Rightarrow$ hyponormal"; the converse is not always true ([5]). Also it holds obviously that "subnormal $\Rightarrow$ polynomially hyponormal $\Rightarrow \cdots \Rightarrow$ weakly 2-hyponormal $\Rightarrow$ hyponormal"; but the converse implications are not developed completely yet except for weak 2-hyponormalities ([11]). In particular, the case in which $k = 2$ has received considerable attention (see, for

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example, [5], [9], [16], [10], [13], [12]), and operators in this class are usually called *quadratically hyponormal.*

On the other hand, the study of *flatness* for weighted shifts is a good approach to detect operator gaps between subnormality and hyponormality ([15], [5], [9], [3], [12]). It was shown in [5] that if W_α is a 2-hyponormal weighted shift with $\alpha_n = \alpha_{n+1}$ for some $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, then $\alpha_1 = \alpha_2 = \cdots$. But in general this flatness does not hold in the case of quadratic hyponormality; for example, if $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{n+1}{n+2}}$ $(n \geq 2)$, then W_α is quadratically hyponormal (cf. [5]). And so the following problem was posed in [6]: *describe all quadratically hyponormal weighted shift with first two weights equal.* In [8], [9] and [16], the recursively generated positively quadratically hyponormal weighted shifts with weight sequence $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ were described, and it was proved that quadratic hyponormality of the weighted shift W_α with $\alpha : \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ is equivalent to positive quadratic hyponormality, where $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ is the Stampfli's subnormal completion of three values $\sqrt{u}$, $\sqrt{v}$, $\sqrt{w}$ (cf. [18]). And also, in [17], the recursively generated positively quadratically hyponormal weighted shifts with weight sequence $\alpha : \sqrt{x}, \sqrt{y}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with Stampfli's tail were described, and it was proved that there exists a quadratically hyponormal but not positively quadratically hyponormal weighted shift W_α. These studies motivated the general back step extension shift W_α with a weight sequence $\alpha : \sqrt{y_m}, \sqrt{y_{m-1}}, \ldots, \sqrt{y_1}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ which is the main model of this paper. As a related study, the following problem, which remains open, was suggested in [9].

Problem 1.1 ([9, Prob. 5.3]). Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 \leq x \leq u \leq v \leq w$. Describe the set of positive real numbers x such that the weighted shift W_α is quadratically hyponormal for some u, v and w.

In this article we solve Problem 1.1 via a characterization of positive quadratic hyponormality.

The organization of this article is as follows. In Section 2 we recall some terminology and notation concerning quadratic hyponormality, positive quadratic hyponormality, and recursively generated weighted shifts, which will be used frequently throughout the paper. In Section 3 we characterize the positive quadratic hyponormality of weighted shifts W_α with weight sequence $\alpha : \sqrt{y_m}, \sqrt{y_{m-1}}, \ldots, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. In Section 4, we review the characterizations of quadratic hyponormality of such weighted shifts W_α, and give a simple proof of Theorem 4.1 in [16] in the case of $m = 1$. In Section 5 we specialize to $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ and apply the general result to characterize the positive quadratic hyponormality of the associated weighted shift W_α. Also, we solve Problem 1.1 via the method for positive quadratic hyponormality. Finally, we obtain a sufficient condition for quadratic hyponormality of weighted shifts and compare positive quadratic hyponormality and quadratic hyponormality: in particular, for the weight sequence $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ for some u, v, w we obtain an interval J in x with nonempty interior in the positive real line such that a weighted shift $W_{\alpha(x)}$ is quadratically hyponormal but not positively quadratically hyponormal on J.

Some of the calculations in this article were aided by use of the software tool *Mathematica* ([19]).

2. Preliminaries and Notation. Let α denote a weight sequence, $\alpha : \alpha_0, \alpha_1, \alpha_2, \ldots$, where it is without loss of generality to assume these are all positive. The weighted shift W_α acting on $\ell^2(\mathbb{N}_0)$, with standard basis $e_0, e_1, \ldots$, is defined by $W_\alpha(e_j) = \alpha_j e_{j+1}$ for all $j \in \mathbb{N}_0$. It is standard that W_α is quadratically hyponormal if and only if $W_\alpha + sW_\alpha^2$ is hyponormal for any $s \in \mathbb{C}$. Let P_n denote the orthogonal projection onto $\vee_{k=0}^n \{e_k\}$. For $s \in \mathbb{C}$ and $n \geq 0$, define

D_n by

$$D_n := D_n(s) = P_n[(W_\alpha + sW_\alpha^2)^*, W_\alpha + sW_\alpha^2]P_n$$
$$= \begin{pmatrix} q_0 & \bar{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \bar{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \bar{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{pmatrix}, \tag{2.1}$$

where $\alpha_{-1} = \alpha_{-2} := 0$ and, for $k \geq 0$,

$$q_k = u_k + |s|^2 v_k, \quad r_k = s\sqrt{w_k}, \quad u_k = \alpha_k^2 - \alpha_{k-1}^2,$$
$$v_k = \alpha_k^2\alpha_{k+1}^2 - \alpha_{k-1}^2\alpha_{k-2}^2, \; w_k = \alpha_k^2(\alpha_{k+1}^2 - \alpha_{k-1}^2)^2, \quad k \geq 0.$$

Since W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$, in anticipation of the usual use of Sylvester's Criterion (which is sometimes called the *Nested Determinant Test*; for example, see [7, p.213]) for positivity of a matrix we consider $d_n(\cdot) := \det(D_n(\cdot))$; d_n is actually a polynomial in $t := |s|^2$ of degree $n+1$, having Maclaurin's expansion

$$d_n(t) := \sum_{i=0}^{n+1} c(n,i)t^i.$$

Since we usually work with these determinants, we abuse notation slightly to regard D_n as a function of $t = |s|^2$ and write, henceforth, $D_n(t)$.

By some standard computations (see, originally, [8]), it is known that

$$d_0 = q_0, \; d_1 = q_0q_1 - |r_0|^2, \; d_{n+2} = q_{n+2}d_{n+1} - |r_{n+1}|^2 d_n, \qquad n \geq 0,$$

and that also

$$\begin{aligned} &c(0,0) = u_0, \; c(0,1) = v_0, \; c(1,0) = u_1u_0, \; c(1,1) = u_1v_0 + u_0v_1 - w_0, \; c(1,2) = v_1v_0, \\ &c(n+2,i) = u_{n+2}c(n+1,i) + v_{n+2}c(n+1,i-1) - w_{n+1}c(n,i-1), \; n \geq 0, \\ &c(n,n+1) = v_0v_1\cdots v_n \geq 0, \; n \geq 0. \end{aligned} \tag{2.2}$$

Note also for future use that u_n, v_n and w_n are non-negative for all $n \geq 0$, at least if we assume (as we shall discuss shortly) the weights are strictly increasing. We will also have frequent occasion to use $z_n := \frac{v_n}{u_n}$ $(n \geq 0)$. If we seek all the d_n non-negative, surely there is an easy way this may occur: in [8] a weighted shift W_α is defined to be *positively quadratically hyponormal* if $c(n,n+1) > 0$ and $c(n,i) \geq 0$, $0 \leq i \leq n$, $n \in \mathbb{N}_0$. This class of operators has been studied in, for example, [5], [16], [1], and [12].

We specialize to a class of shifts due to Stampfli and arising in his consideration of the problem of completing an initial finite sequence of weights to yield the weight sequence for a subnormal shift ([18]). Given weights $0 < \alpha_0 < \alpha_1 < \alpha_2$ there is a canonical way to generate a satisfactory completion recursively: define

$$\widehat{\alpha}_n = \left(\Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2}\right)^{1/2}, \qquad n \geq 3,$$

where

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2}, \qquad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}.$$

This produces a bounded sequence $\widehat{\alpha} := \{\widehat{\alpha}_i\}_{i=0}^{\infty}$, for which $\widehat{\alpha}_i = \alpha_i$ $(0 \le i \le 2)$. We usually write the weight sequence $(\alpha_0, \alpha_1, \alpha_2)^\wedge$ and resulting weighted shift as $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$.

It has been standard in the study of the weaker than subnormal classes to consider backstep extensions of (or perturbations of) subnormal weighted shifts. Given some weight sequence $\alpha : \alpha_0, \alpha_1, \ldots$ yielding a subnormal shift, prefix some weights (viewed as parameters) and consider the ranges of the parameters yielding some property of interest. The "length" m of the backstep extension is the number of prefixed weights. A weighted shift is hyponormal if and only if its weight sequence is weakly increasing. It therefore follows from various "flatness" results (in which, under most conditions, a pair of equal weights forces all or almost all weights to be equal – see, for example, [5]) that is sufficient to consider strictly increasing sequences of weights.

Throughout this paper $\mathbb{R}_0$ is the set of nonnegative real numbers.

3. Positive Quadratic Hyponormality. We turn first to positive quadratic hyponormality of certain backstep extensions; the results here generalize to arbitrary length extensions, and regularize the somewhat *ad hoc* computations of, [16], [12]. As promised we specialize to an m-length backstep extension of a (normalized) Stampfli weight sequence and its associated shift: denote the weight sequence by

$$\sqrt{y_m}, \sqrt{y_{m-1}}, \ldots, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge.$$

It turns out that the matrix $D_{m+1}(t)$ is particularly important both for quadratic hyponormality and positive quadratic hyponormality, and we begin with it, a variant, and some elementary conclusions. Let $\hat{D}_n(t)$ be the matrix obtained by subtracting u_n from the bottom right entry; that is,

$$\hat{D}_n(t) = D_n(t) - \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & u_n \end{pmatrix}.$$

It is elementary that $\det(\hat{D}_n(t)) = \det(D_n(t)) - u_n \det(\hat{D}_{n-1}(t))$, and therefore that if we write $\det(\hat{D}_n(t)) = \sum_{i=0}^{n+1} \hat{c}(n,i)t^i$, then

$$\hat{c}(n, n+1) = c(n, n+1), \quad \hat{c}(n,0) = c(n,0) - u_n c(n-1,0) = 0,$$

and further for $1 \le i \le n$, and $n \ge 2$,

$$\begin{aligned} \hat{c}(n,i) &= c(n,i) - u_n c(n-1,i) \\ &= u_n c(n-1,i) + v_n c(n-1,i-1) - w_{n-1} c(n-2,i-1) - u_n c(n-1,i) \\ &= v_n c(n-1,i-1) - w_{n-1} c(n-2,i-1), \end{aligned}$$

where we have used the recursion in (2.2). In particular,

$$\det(\hat{D}_{m+1}(t)) = \sum_{i=0}^{m+2} \hat{c}(m+1,i)t^i$$

with $\hat{c}(m+1, m+2) = c(m+1, m+2)$, $\hat{c}(m+1,0) = 0$, and

$\hat{c}(m+1,i) = v_{m+1} c(m,i-1) - w_m c(m-1,i-1), \quad 1 \le i \le m+1.$

It is the $\hat{c}(m+1,i)$ that have appeared, unrecognized and in a plethora of notations, in prior computations for special cases of backstep extensions.

It is known from [9] that for an m-length backstep extension of a Stampfli weight sequence one has $u_n v_{n+1} - w_n = 0$ for $n \geq m+1$. This fact, plus an elementary matrix computation expanding $\det(D_n(t))$ by its last row, yields

$$\det(D_n(t)) = u_n \det(D_{n-1}(t)) + v_n t \det(\hat{D}_{n-1}(t)).$$

An induction then produces, for $n \geq m+2$,

$$\det(D_n(t)) = u_n \det(D_{n-1}(t)) + v_n \cdots v_{m+2} t^{n-(m+1)} \det(\hat{D}_{m+1}(t)).$$

At the coefficient level this becomes, for $n \geq m+2$,

$$c(n,i) = \begin{cases} u_n c(n-1,n) + v_n \cdots v_{m+2} \left(\hat{c}(m+1, i-(n-m)+1)\right), \\ \qquad\qquad\qquad\qquad n-m \leq i \leq n+1, \\ u_n c(n-1,i), \qquad 0 \leq i \leq n-(m+1). \end{cases} \tag{3.1}$$

It is well-known that $c(n,n+1) = v_0 v_1 \cdots v_n \geq 0$ for all $n \in \mathbb{N}_0$ (see [1]), so there is simplification for the term with $i = n+1$ in the above. The relations in (3.1) have been obtained for one- and two-length backstep extensions in [16] and [12] respectively.

Further computations (recalling $z_n = v_n/u_n$) show

$$c(n,n) = \begin{cases} \frac{v_{m+2} \cdots v_{n-1}}{u_n} \left(v_0 \cdots v_{m+1} + z_n \hat{c}(m+1,m+1)\right), & n > m+2, \\ v_0 \cdots v_{m+1} + z_n \hat{c}(m+1,m+1), & n = m+2. \end{cases}$$

Therefore, since the v_j and u_j are non-negative, the sign of $c(n,n)$, for $n \geq m+2$, reduces to that of $v_0 \cdots v_{m+1} + z_n \hat{c}(m+1,m+1)$. Similar computations, and a finite induction, show that for $n \geq m+2+i$ and $0 \leq i \leq m$, the sign of $c(n,n-i)$ is the same as that of

$$\begin{aligned} G(n,i) &:= G(n,i,z_{n-i},\ldots,z_n) \\ &:= v_0 \cdots v_{m+1} + \sum_{k=0}^{i} (\Pi_{j=0}^{k} z_{n-i+j}) \hat{c}(m+1, m-k+1) \\ &= \hat{c}(m+1,m+2) + \sum_{k=0}^{i} (\Pi_{j=0}^{k} z_{n-i+j}) \hat{c}(m+1, m-k+1), \end{aligned}$$

where we have used a standard formula for $c(n,n+1)$ to obtain the last equality. Productive analysis of these signs will hinge on considering differences; define

$$Q(k,n) := \frac{\Pi_{i=0}^{k-1} z_{n+i} - \Pi_{i=-1}^{k-2} z_{n+i}}{z_n - z_{n-1}}, \qquad n \geq 1, \ k \geq 1.$$

Define next ΔG by

$$\Delta G(n,i) := G(n+1,i,z_{n-i+1},\ldots,z_{n+1}) - G(n,i,z_{n-i},\ldots,z_n),$$

for $n \geq m+2+i, 0 \leq i \leq m$. It is easy to see that for $n \geq m+2+i$ and $0 \leq i \leq m$,

$$\Delta G(n,i) = (z_n - z_{n-1}) \cdot \left(\sum_{k=0}^{i} Q(k+1,n)\, \hat{c}(m+1, m-k+1) \right). \tag{3.2}$$

It is known that the z_n increase strictly (at least for $n \geq m+2$), so $\Delta G(n,i)$ will be positive or negative exactly as the second term above, and thus $G(n,i)$ will be increasing, or decreasing, exactly with the sign of the second term above.

Proposition 3.1. *Consider the usual recursively generated weight sequence* $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. *Then, with A and B as in Lemma 3.1 and $Q(k,n)$ as above,*

$$Q(k+2,n) = A \cdot Q(k-1,n) + B \cdot Q(k+1,n), \qquad k \geq 2, n \geq 3,$$

where $A = -\frac{v^4(u-w)^5}{u(u-v)^4(v-w)}$ *and* $B = \frac{v(u-w)(uv(v-3w)+u^2w+vw^2)}{u(u-v)^2(v-w)}$. *In particular, for any $k \geq 1$, $Q(k,n)$ is a constant in u, v, and w for all $n \geq 3$. Further, if we consider instead the $Q(k,n)$ associated with an m-length backstep extension of* $(\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$, *one has for any $k \geq 1$ that $Q(k,n)$ is a constant in u, v, and w for any $n \geq m+3$. Note also that $Q(1,n)$ is trivially constant in n.*

Return now to the second term arising from $\Delta G(n,i)$ in (3.2), and for $n \geq m+2+i, 0 \leq i \leq m$. Since m is constant, surely the relevant $\hat{c}(m, m-i)$ are constants in the data $y_m, \ldots, y_1$, u, v, and w. Using Proposition 3.1 the $Q(k,n)$ are as well. Thus, for any $0 \leq i \leq m$, $\Delta G(n,i)$ is either all positive for $n \geq m+2+i$ or all negative for $n \geq m+2+i$. Therefore, $G(n,i)$ is either (all) increasing for $n \geq m+2+i$ or (all) decreasing for $n \geq m+2+i$. But $G(n,i)$ controls the sign of $c(n, n-i)$. If $G(n,i)$ is all increasing, for positivity of all the $c(n,n-i)$ for $n \geq m+2+i$ it is enough to check the sign of $G(m+2+i, i)$ or equivalently that of $c(m+2+i, m+2)$. If $G(n,i)$ is decreasing, we must check the limit as n becomes large. It is well-known that the z_n increase to a limit, and that in fact

$$\begin{aligned} K = \lim_{n\to\infty} z_n &= \frac{-\Psi^2}{2\Psi_0}(\Psi_1 + \sqrt{\Psi_1^2 + 4\Psi_0}) \\ &= \frac{v(w-u)^2 \left(v(w-u) + \sqrt{v^2(w-u)^2 - 4uv(w-v)(v-u)}\right)}{2u(v-u)^2(w-v)}. \end{aligned} \tag{3.3}$$

Surely $G(n,i)$ is a continuous function of its z_j. Thus in the case of $G(n,i)$ increasing in n, we must check the sign of the expression $G(n,i,K,\ldots,K)$, where

$$G(n,i,K,\ldots,K) = v_0 \cdots v_{m+1} + \textstyle\sum_{k=0}^{i} K^{k+1} \hat{c}(m+1, m-k+1),$$

which comes from $G(n,i,z_{n-i},\ldots,z_n)$ as $z_n \nearrow K$. Putting this together and recalling the expression for $\Delta G(n,i)$ from (3.2) we obtain the following.

Proposition 3.2. *Let* $\alpha : \sqrt{y_m}, \sqrt{y_{m-1}}, \ldots, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ *be the usual m-length backstep extension of a Stampfli weight sequence. Let i satisfying $0 \leq i \leq m$ be fixed. Then for the associated shift W_α, $c(n, n-i) \geq 0$ for all $n \geq m+2+i$ if and only if either*

1° $\sum_{k=0}^{i} Q(k+1, m+3)\, \hat{c}(m+1, m-k+1) \geq 0$ *(the increasing case) and* $c(m+2+i, m+2) \geq 0$

or

2° $\sum_{k=0}^{i} Q(k+1, m+3)\, \hat{c}(m+1, m-k+1) \leq 0$ *(the decreasing case) and*
$v_0 \cdots v_{m+1} + \sum_{k=0}^{i} K^{k+1} \hat{c}(m+1, m-k+1) \geq 0.$

Note that it is clear that if for some i we are in the "increasing" case 1°, then even if not all the $c(n, n-i)$ are non-negative for $n \geq m+2+i$, if for some ℓ it happens that $c(\ell, \ell-i) \geq 0$, then for all $j \geq 0$, $c(\ell+j, \ell+j-i) \geq 0$. Equally easy is that if we are in the decreasing case 2° and $G(n,i,K,\ldots,K) < 0$, then eventually all the $c(n,n-i)$ are negative if n is large enough.

From the second term in (3.1) it is easy to show that if for some p, $1 \leq p \leq m+1$, $c(m+p, p) \geq 0$, then for all $j \geq 0$, $c(m+p+j, p) \geq 0$. As well, if for all $p \geq 0$, $c(2m+2+p, m+2+p) \geq 0$, then for all $j \geq 0$ and all $p \geq 0$, $c(2m+2+p+j, m+2+p) \geq 0$. It is known that all the $c(n, n+1)$ and $c(n, 0)$ are non-negative, and easy computation shows that $c(j,k) \geq 0$ for $j = 0, 1, 2$ and the relevant k. Assembling this all gives finally the following.

Theorem 3.3. *Consider the usual m-length backstep extension of a Stampfli sequence, $m \geq 1$, with resulting sequence*

$$\alpha : \sqrt{y_m}, \sqrt{y_{m-1}}, \ldots, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge.$$

The associated weighted shift W_α is positively quadratically hyponormal if and only if the following conditions hold:

1° *for each* i, $0 \le i \le m$, *one of the conditions of Proposition 3.2 holds,*

2° *for each* n, $3 \le n \le m+1$, *and each* j, $1 \le j \le n$, $c(n,j) \ge 0$,

3° *for each* n, $m+2 \le n \le 2m+1$, *and each* j, $n-m \le j \le m+1$, $c(n,j) \ge 0$.

(*Observe that if* $m = 1$, *the second of these conditions is vacuous.*)

It is enlightening to compare this result with that of Theorem 3.7 from [1] in which, in the presence of some stronger conditions not satisfied by a recursively generated weighted shift, what must be checked is essentially numbers 2° and 3° of the above list.

4. Quadratic Hyponormality. In this section we briefly recast, in more useful notation, and generalize somewhat, the discussion from [16] concerning quadratic hyponormality for backstep extensions of Stampfli shifts. Recall that quadratic hyponormality of some weighted shift W_α is equivalent to the positivity of the matrices $D_n(s)$ for all $s \in \mathbb{C}$, as in the introduction. As usual for each n this is equivalent to the positivity of the associated complex quadratic form. The consequence of [16] Lemma 3.1 is that this is equivalent to the positivity of a real quadratic form associated with a slightly different matrix. Define, for $t \ge 0$, and for each n,

$$A_n(t) = \begin{pmatrix} q_0 & -\sqrt{w_0 t} & 0 & \cdots & 0 & 0 \\ -\sqrt{w_0 t} & q_1 & -\sqrt{w_1 t} & \cdots & 0 & 0 \\ 0 & -\sqrt{w_1 t} & q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & -\sqrt{w_{n-1} t} \\ 0 & 0 & 0 & \cdots & -\sqrt{w_{n-1} t} & q_n \end{pmatrix}, \tag{4.1}$$

where the q_i, u_i, v_i, and w_i are as for D_n as in (2.1). Then from Lemma 3.1 of [16] one obtains

Proposition 4.1. *Let W_α be a weighted shift. Then for each n, $D_n(s)$ is a (complex) positive quadratic form over $\mathbb{C}^{n+1}$ for all $s \in \mathbb{C}$ if and only if $A_n(t)$ is a (real) positive quadratic form over $\mathbb{R}^{n+1}$ for all $t \ge 0$.*

Specialize now to the case of the m-length backstep extension of a Stampfli shift with weight sequence $\sqrt{y_m}, \sqrt{y_{m-1}}, \dots, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. We will follow the development in Sections 3 and 4 of [16], suitably generalized from a 2-length backstep extension to an m-length backstep extension. To test positivity of the quadratic form for $A_n(t)$ at a vector $(x_0, \dots, x_n)$ in $\mathbb{R}_0^{n+1}$, what arises is

$$F_n(x_0, \dots, x_n, t) := \sum_{i=0}^{n} u_i x_i^2 - 2\sqrt{t} \sum_{i=0}^{n-1} \sqrt{w_i} x_i x_{i+1} + t \sum_{i=0}^{n} v_i x_i^2.$$

It will be important, in considering positivity for $n \ge m+1$, to split off a piece of this expression. Namely, for $n \ge m+1$ let f be defined by

$$f(x_0, \dots, x_n, t) := \sum_{i=0}^{m} u_i x_i^2 - 2\sqrt{t} \sum_{i=0}^{m-1} \sqrt{w_i} x_i x_{i+1} + t \sum_{i=0}^{m+1} v_i x_i^2.$$

It follows that for $n \ge m+1$,

$$\begin{aligned} F_n(x_0, \dots, x_n, t) &= f(x_0, \dots, x_{m+1}, t) + \sum_{i=m+1}^{n} u_i x_i^2 - 2\sqrt{t} \sum_{i=0}^{n-1} \sqrt{w_i} x_i x_{i+1} + t \sum_{i=0}^{n} v_i x_i^2 \\ &= f(x_0, \dots, x_{m+1}, t) + \sum_{i=m+1}^{n-1} (\sqrt{u_i} x_i - \sqrt{v_{i+1} t}\, x_{i+1})^2 + u_n x_n^2, \end{aligned} \tag{4.2}$$

where the final equality is the result of a simple computation and the use of $u_i v_{i+1} = w_i$ for $i \geq m+2$.

Recall that $K = \lim_{n\to\infty} z_n$ with explicit expression as in (3.3). Continuing the argument as in [16] we arrive at the following, which is the analog of their Lemma 4.4.

Proposition 4.2. *For the usual weight sequence*

$$\sqrt{y_m}, \sqrt{y_{m-1}}, \ldots, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$$

and its associated weighted shift W_α, the following assertions are equivalent:

(i) $F_n(x_0, \ldots, x_n, s) \geq 0$ *for any* $x_0, \ldots, x_n, s$ *in* $\mathbb{R}_0$ *and* $\sqrt{\frac{1}{K}} < s$ *for all* $n \geq m+2$,

(ii) $f(x_0, \ldots, x_{m+1}, s) \geq 0$ *for all* $x_0, \ldots, x_{m+1}, s$ *in* $\mathbb{R}_0$ *and* $\sqrt{\frac{1}{K}} < s$.

Equivalently, in terms of $t = s^2$, the following assertions are equivalent:

(i′) $F_n(x_0, \ldots, x_n, t) \geq 0$ *for any* $x_0, \ldots, x_n, t$ *in* $\mathbb{R}_0$ *and* $\frac{1}{K} < t$ *for all* $n \geq m+2$

(ii′) $f(x_0, \ldots, x_{m+1}, t) \geq 0$ *for all* $x_0, \ldots, x_{m+1}, t$ *in* $\mathbb{R}_0$ *and* $\frac{1}{K} < t$.

It is obvious that if some F_{n+1} is appropriately positive then so is F_n, and we have the following generalization of [16, Th. 4.6].

Theorem 4.3. *For the usual weight sequence*

$$\sqrt{y_m}, \sqrt{y_{m-1}}, \ldots, \sqrt{y_1}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge,$$

the following assertions are equivalent:

(i) *the associated weighted shift W_α is quadratically hyponormal,*

(ii) *the following two conditions hold:*

1° *for each* $n \geq 3$, *each* t *in* $[0, \frac{1}{K}]$, *and each* $x_0, \ldots, x_n$ *in* $\mathbb{R}_0$, $F_n(x_0, \ldots, x_n, t) \geq 0$,

2° *for all* $x_0, \ldots, x_{m+1}$ *in* $\mathbb{R}_0$ *and all* $\frac{1}{K} < t$, $f(x_0, \ldots, x_{m+1}, t) \geq 0$.

5. Quadratic Hyponormality for Weights $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$

In this section we consider the weighted shifts W_α with a weight sequence $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ satisfying $1 < x < u < v < w$; for example, see Problem 1.1. The following lemma was obtained by rearranging indices in some of the results of [8, p.394].

Lemma 5.1 ([9, Lemma 2.1]). *Let* $\alpha : \alpha_0, \alpha_1, \cdots, \alpha_{k-2}, (\alpha_{k-1}, \alpha_k, \alpha_{k+1})^\wedge$ *with* $0 < \alpha_{k-1} < \alpha_k < \alpha_{k+1}$ $(k \geq 1)$ *and let W_α be the unilateral weighted shift associated to α. Then*

(i) $v_{n+1} = \Psi_1(u_{n+1} + u_n) \quad (n \geq k)$

(ii) $w_n = u_n v_{n+1}$ $(n \geq k)$ *and*

$$u_n = -\Psi_0 \frac{u_{n-1}}{\alpha_{n-2}^2 \alpha_{n-1}^2} \quad (n \geq k+1), \tag{5.1}$$

where

$$\Psi_0 = -\frac{\alpha_k^2 \alpha_{k-1}^2 (\alpha_{k+1}^2 - \alpha_k^2)}{\alpha_k^2 - \alpha_{k-1}^2} \quad \textit{and} \quad \Psi_1 = \frac{\alpha_k^2 (\alpha_{k+1}^2 - \alpha_{k-1}^2)}{\alpha_k^2 - \alpha_{k-1}^2}. \tag{5.2}$$

As noted before, to avoid trivialities we may and do assume that $\alpha_k < \alpha_{k+1}$ for all $k \in \mathbb{N}$ to avoid the trivial case throughout this section.

5.1 Positively quadratic hyponormality. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with real variables x, u, v, w satisfying $1 < x < u < v < w$. For our convenience, we recall from (5.2) that

$$\Psi_1 = \frac{v(w-u)}{v-u} \quad \text{and} \quad \Psi_0 = -\frac{uv(w-v)}{v-u}.$$

Before we begin our work, we compute directly the coefficients of $d_n(t)$ for the reader's convenience, since having these concretely will simply some arguments in what follows:

$$\begin{aligned}
&c(0,0)=1,\quad c(1,0)=0,\quad c(2,0)=0,\quad c(3,0)=0,\quad c(4,0)=0,\\
&c(0,1)=1,\quad c(1,1)=x-1,\quad c(2,1)=0,\quad c(3,1)=0,\quad c(4,1)=0,\\
&c(1,2)=x,\ c(2,2)=ux(x-1),\ c(3,2)=x(x-1)(2u-1-ux),\ c(4,2)=(v-u)c(3,2),\\
&c(2,3)=x(ux-1),\quad c(3,3)=ux(x-1)(uv+1-2x),\quad c(4,3)=x(x-1)f_1(x),\\
&c(3,4)=(uv-x)x(ux-1),\quad c(4,4)=xf_2(x)\ \text{ and }\ c(4,5)=v_4c(3,4),
\end{aligned}$$

where

$$\begin{aligned}
f_1(x)&=-(2u^2v-2uv-uvw+u)x-(u^3v+u^2+vw-uv-2uvw),\\
f_2(x)&=u^2x^3+(-uv-2uvw-u^3v-2u^2+2u^2v+u)x^2\\
&\quad+(v-u+u^2v^2w+3uvw+u^2-2uv)x+(u^2v-uvw-u^2v^2w).
\end{aligned}$$

The following lemma converts the general equations in (2.2) to the special case of the weights considered here. Note also that $\eta_2 = v_4c(3,3) - w_3c(2,3)$ is just $\hat{c}(4,4)$ and $\eta_3 = v_4c(3,2) - w_3c(2,2)$ is just $\hat{c}(4,3)$; that is, the results for terms like $c(n,i)$ are available from the general framework of Section 3, but it is somewhat less cumbersome to compute directly.

Lemma 5.2. *Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. Then for $n \geq 5$, we have*

$$c(n,i)=\begin{cases}
v_n\cdots v_4c(3,4), & i=n+1,\\
u_nc(n-1,n)+v_n\cdots v_5\eta_2, & i=n,\\
u_nc(n-1,n-1)+v_n\cdots v_5\eta_3, & i=n-1,\\
u_n\cdots u_{i+2}c(i+1,i), & 3\le i\le n-2,\\
u_n\cdots u_5c(4,i), & 0\le i\le 2,
\end{cases}$$

where $\eta_2 = v_4c(3,3) - w_3c(2,3)$ and $\eta_3 = v_4c(3,2) - w_3c(2,2)$.

We now characterize the positive quadratic hyponormality of the weighted shift W_α with weight sequence $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$.

Theorem 5.3. *Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. Under the above notation, it holds that W_α is positively quadratically hyponormal if and only if*

(i) $c(3,2) \geq 0$ (*i.e.*, $x \leq 2 - \frac{1}{u}$),

(ii) $c(4,3) \geq 0$, $c(4,4) \geq 0$ (*i.e.*, $f_1(x) \geq 0$, $f_2(x) \geq 0$),

(iii) $c(5,5) \geq 0$, $c(5,4) \geq 0$,

(iv) *it holds that*

  1° $\eta_2 \geq 0$, or

  2° $(\eta_2 < 0)$ *and* $K \leq \frac{\eta_1}{|\eta_2|}$;

(v) *it holds that*

  1° *if* $\eta_2 + \frac{z_7z_6 - z_6z_5}{z_6 - z_5}\eta_3 \geq 0$ *then* $\eta_1 + z_5\eta_2 + z_5z_6\eta_3 \geq 0$ *or equivalently*, $c(6,5) \geq 0$;

  2° *if* $\eta_2 + \frac{z_7z_6 - z_6z_5}{z_6 - z_5}\eta_3 < 0$ *then* $\eta_1 + \eta_2K + \eta_3K^2 \geq 0$.

We now discuss an example using Theorem 5.3.

Example 5.4. Let $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. Take $u = \frac{11}{10}$, $v = \frac{12}{10}$, and $w = \frac{15}{10}$. By some computations, we get (with only possibly relevant ranges presented):

(i) $x \leq 2 - \frac{1}{u} \Longleftrightarrow 1 < x \leq \frac{12}{11} \approx 1.09090\cdots$,

(ii) $f_1(x) \geq 0,\ f_2(x) \geq 0 \Longleftrightarrow 1.00462\cdots \leq x \leq \frac{841}{770} \approx 1.09220\cdots$,

(iii) $c(5,5) \geq 0,\ c(5,4) \geq 0 \Longleftrightarrow 1.00349\cdots \leq x \leq 1.09629\cdots$.

Note that $c(6,5) \geq 0$ for $1.022\cdots \leq x \leq 1.09312\cdots$ should it be required for (v). We now consider the following inequalities restricting our attention to only x with $1.00462\cdots \leq x \leq \frac{12}{11}$.

(iv) Observe that $\eta_2 < 0 \Leftrightarrow 1.00462\cdots \le x < 1.02095\cdots$. In this range, we get $\eta_1+K\eta_2 \ge 0$ for $1.01499\cdots \le x < 1.02095\cdots$ (and $\eta_1 + K\eta_2 < 0$ for $1.00462\cdots \le x < 1.01499\cdots$). And observe that $\eta_2 \ge 0 \Leftrightarrow 1.02095\cdots \le x \le 1.09090\cdots$. Hence (iv) holds $\Leftrightarrow 1.01499\cdots \le x \le 1.09090\cdots$.

(v) Observe that $\eta_2 + \frac{z_7z_6-z_6z_5}{z_6-z_5}\eta_3 \ge 0 \Leftrightarrow 1.00568\cdots \le x \le 1.06172\cdots$. As noted above, $c(6,5) \ge 0$ for $1.022\cdots \le x \le 1.09312\cdots$ And observe that, under the range of x with $1.00462\cdots \le x \le \frac{12}{11}$, $\eta_2 + \frac{z_7z_6-z_6z_5}{z_6-z_5}\eta_3 < 0 \Leftrightarrow 1.00462\cdots \le x < 1.00568\cdots$ and $1.06172\cdots < x \le \frac{12}{11}$. And we obtain that $\eta_1 + \eta_2 K + \eta_3 K^2 \ge 0$ for $1.00505\cdots \le x < 1.06584\cdots$; in our region of interest, the expression is negative to the left and right of these bounds. So (v) holds $\Leftrightarrow 1.00505\cdots \le x \le 1.06583\cdots$.

Combining (i)-(v), we have that W_α is positively quadratically hyponormal $\Leftrightarrow \delta_1 = 1.01499\cdots \le x \le \delta_2 = 1.06583\cdots$, where δ_1 is the smallest positive root of $\eta_1 + K\eta_2 = 0$, where

$$\eta_1 + K\eta_2 = \tfrac{x}{12500}\big((52800\sqrt{5}+226325)x^3 - (226176\sqrt{5}+963169)x^2 +(304512\sqrt{5}+1291368)x - 131328\sqrt{5} - 555012\big),$$

and δ_2 is the second smallest positive root of $\eta_1 + \eta_2 K + \eta_3 K^2 = 0$, where

$$\eta_1 + \eta_2 K + \eta_3 K^2 = \tfrac{x}{7562500}\big((136926625 + 31944000\sqrt{5})x^3 - (1349119805 +428799360\sqrt{5})x^2 + (2355886536 + 784080768\sqrt{5})x - (1143988596 + 387341568\sqrt{5})\big).$$

5.2. A solution of Problem 1.1. In this subsection we solve Problem 1.1 by characterizing the quadratically and positively quadratically hyponormal weighted shifts which are three-length backstep extensions of the Stampfli recursive tail, and with the first two weights equal. As noted before, by scaling it suffices to consider the case in which the first two weights are one.

Theorem 5.5. *Let $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ be a weight sequence with positive real variables x, u, v, w such that with $1 < x < u < v < w$. Let*

(PQH) $:= \{x | W_{\alpha(x)}$ *is positively quadratically hyponormal for some u, v and w*$\}$

and

(QH) $:= \{x | W_{\alpha(x)}$ *is quadratically hyponormal for some u, v and w*$\}$.

Then (PQH) $=$ (QH) $= (1, 2)$.

In fact, we have the following a priori stronger result.

Theorem 5.6. *Let δ be an arbitrary real number in the open interval $(1, 2)$. Then there exist $u_\delta, v_\delta, w_\delta$ with $u_\delta < v_\delta < w_\delta$ such that $W_{\alpha(x)}$ is positively quadratically hyponormal weighted shift with a weight sequence $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u_\delta}, \sqrt{v_\delta}, \sqrt{w_\delta})^\wedge$ for all $x \in (1, \delta]$.*

5.3. Quadratic Hyponormality. In this subsection we discuss gaps between the quadratic and positive quadratic hyponormalities. For given u, v, w with $1 < u < v < w$, let $\alpha := \alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u$. Then it is worthwhile to obtain an interval J with nonempty interior in $\mathbb{R}_0$ such that $W_{\alpha(x)}$ is quadratically hyponormal but not positively quadratically hyponormal for all $x \in J$. We shall obtain such a nonempty interval in Example 5.8 below.

Consider a weight sequence $\alpha : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ with $1 < x < u < v < w$. Since $w_n = u_n v_{n+1}$ $(n \ge 4)$, we get

$$F_n := F_n(x_0, \cdots, x_n, s) = f(x_0, x_1, ..., x_4, t) + \textstyle\sum_{i=4}^{n-1}(\sqrt{u_i}x_i - \sqrt{v_{i+1}}sx_{i+1})^2 + u_n x_n^2,$$

where $t := |s|^2$ and

$$\begin{aligned} f(x_0,x_1,x_2,x_3,x_4,t) &= (u_0 + tv_0)\,x_0^2 + (u_1 + tv_1)\,x_1^2 + (u_2 + tv_2)\,x_2^2 \\ &+ (u_3 + tv_3)\,x_3^2 - 2s(\sqrt{w_0}x_0x_1 + \sqrt{w_1}x_1x_2 + \sqrt{w_2}x_2x_3 + \sqrt{w_3}x_3x_4) + tv_4x_4^2. \end{aligned}$$

It follows from Theorem 4.3 and (4.2) that if $f(x_0, x_1, x_2, x_3, x_4, t) \geq 0$, then W_α is quadratically hyponormal. Observe that the quadratic function of $f(x_0, x_1, x_2, x_3, x_4, t)$ corresponds to the symmetric matrix

$$\Delta(t) = \begin{pmatrix} u_0 + v_0 t & -\sqrt{w_0 t} & 0 & 0 & 0 \\ -\sqrt{w_0 t} & u_1 + v_1 t & -\sqrt{w_1 t} & 0 & 0 \\ 0 & -\sqrt{w_1 t} & u_2 + v_2 t & -\sqrt{w_2 t} & 0 \\ 0 & 0 & -\sqrt{w_2 t} & u_3 + v_3 t & -\sqrt{w_3 t} \\ 0 & 0 & 0 & -\sqrt{w_3 t} & v_4 t \end{pmatrix};$$

also, see (4.1). Hence it follows from the positivity $\Delta(t) \geq 0$ that W_α is quadratically hyponormal. By some computations, $\Delta(t)$ can be represented by the matrix

$$\begin{pmatrix} 1+t & -\sqrt{t} & 0 & 0 & 0 \\ -\sqrt{t} & tx & -\sqrt{t}(x-1) & 0 & 0 \\ 0 & -\sqrt{t}(x-1) & -1+x+t(-1+ux) & -\sqrt{tx}(u-1) & 0 \\ 0 & 0 & -\sqrt{tx}(u-1) & u+t(uv-x)-x & -\sqrt{ut}(v-x) \\ 0 & 0 & 0 & -\sqrt{ut}(v-x) & t(vw-ux) \end{pmatrix}.$$

Recall the upper left $n \times n$ proper submatrices of $\Delta(t)$ are exactly $A_{n-1}(t)$, $n = 1, 2, 3, 4$, which are as in (4.1).

Proposition 5.7. *Let* $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$ *with* $1 < x < u < v < w$. *If* $\phi_1(t) := c(3,4)t^2 + c(3,3)t + c(3,2) > 0$ and $\phi_2(t) := \eta_1 t^2 + \eta_2 t + \eta_3 \geq 0$ *for all* $t > 0$, *then* $W_{\alpha(x)}$ *is quadratically hyponormal.*

Example 5.8 (Continued from Example 5.4). Let $\alpha(x) : 1, 1, \sqrt{x}, (\sqrt{u}, \sqrt{v}, \sqrt{w})^\wedge$. Take $u = \frac{11}{10}$, $v = \frac{12}{10}$, and $w = \frac{15}{10}$. By some direct computations, we obtain that

$\eta_1 = \frac{x}{2500}\left(-5940 + 14664x - 11693x^2 + 3025x^3\right)$,
$\eta_2 = \frac{x}{2500}\left(-7524 + 17446x - 12958x^2 + 3025x^3\right)$ and
$\eta_3 = \frac{x}{2500}\left(-1044 + 2034x - 990x^2\right)$.

If one of the following two cases holds:
(i) $\eta_2, \eta_3 \geq 0$ (note: $\eta_1 > 0$),
(ii) $\eta_3 \geq 0$ and the discriminant $\eta_2^2 - 4\eta_1\eta_3 \leq 0$,
then $\phi_2(t) \geq 0$ for all $t \geq 0$ and all x satisfying (i)-(ii). By direct computations, we obtain the range $1.00544\cdots \leq x \leq 1.05454\cdots$ for x satisfying (i)-(ii).

Similarly, applying $\phi_1(t)$ with the same method above, we obtain a range $1 < x < \frac{12}{11}$ for $\phi_1(t) \geq 0$ for all t.

According to Proposition 5.7, if we take x satisfying $1.00544\cdots \leq x \leq 1.05454\cdots$, then W_α is quadratically hyponormal. Hence, with the range of x for the positive quadratic hyponormality in Example 5.4, we obtain a range $1.00544\cdots \leq x \leq 1.06583\cdots$ for the quadratic hyponormality of $W_{\alpha(x)}$. Thus the open interval $\left(\frac{100545}{100000}, \frac{10149}{10000}\right)$ is a range in x for quadratic hyponormality but not positive quadratic hyponormality of $W_{\alpha(x)}$.

Observe finally that the test in Proposition 5.7 is clearly sufficient, but not necessary, for quadratic hyponormality, as evidenced by the interval $1.05454\cdots \leq x \leq 1.06583\cdots$.

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